Name:________________________________ Wed - Quiz on Washers/Disks 10/4

Lessons 17 and 18 - Solids of Reviolution - Shells
Math 16020

III Washers vs. Shells

So far, we have used disks and washers to find volumes. A washer is like a washer that you would see in a hardware store. It is very short, but the difference between the two radii can be significant.

 dy disks and washers stack.

Today we are going to use **shells** to find volumes. Shells are like a soup can that has no top or bottom. They can be tall, but the difference between the two radii is quite small.

dx

shells expand.

Whether we are working with washers or shells, it is important to remember that:

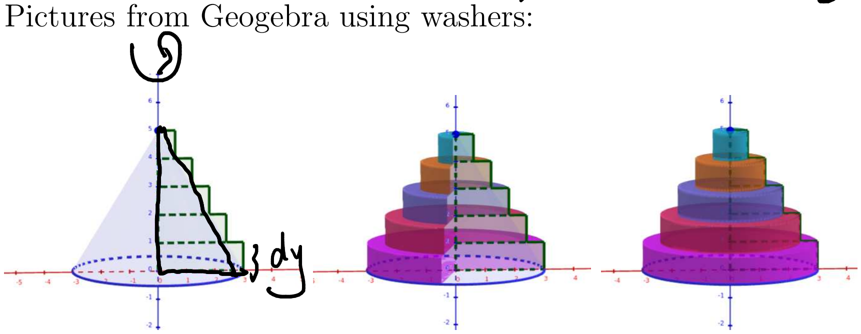
- The **centers** of the circles are at the **axis of rotation**.
- The **radii** of the circles extend **out from the axis of rotation.**

radii are perpendicular ($\perp$) to axis of rotation.

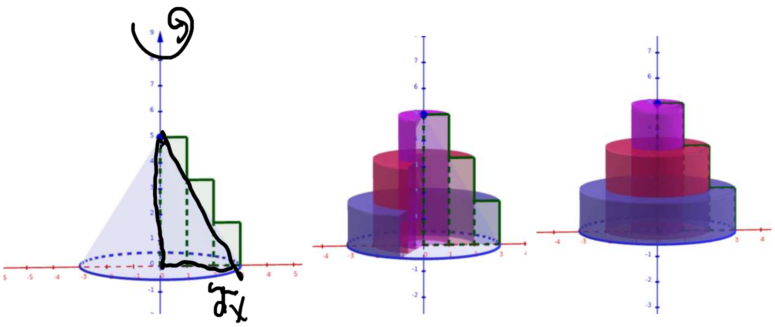
Let's take a look at the cone that we created in a previous class by rotating the triangle with vertices $(0,0)$, $(3,0)$, and $(0,5)$ about the y-axis.

disks (no hole)

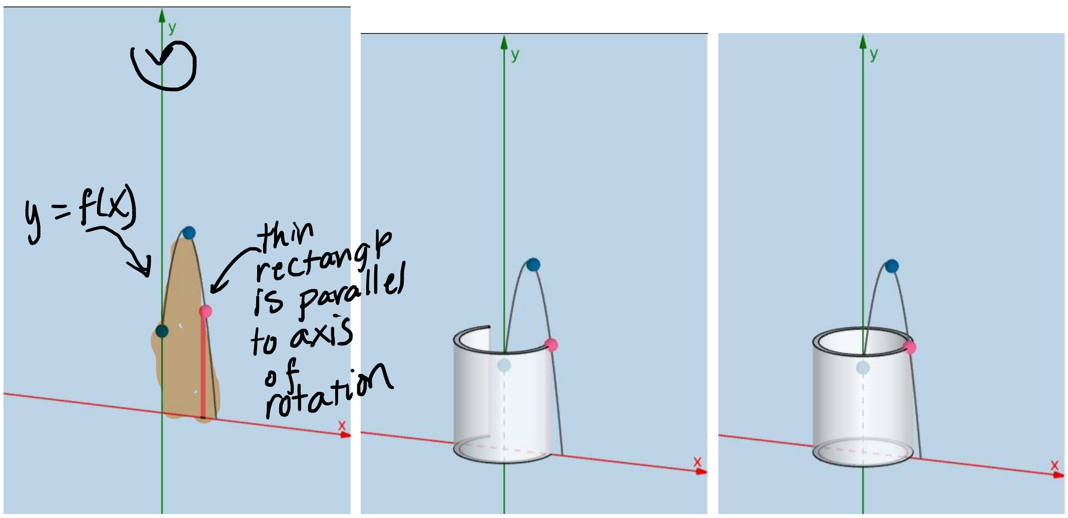
Pictures from Geogebra using washers:

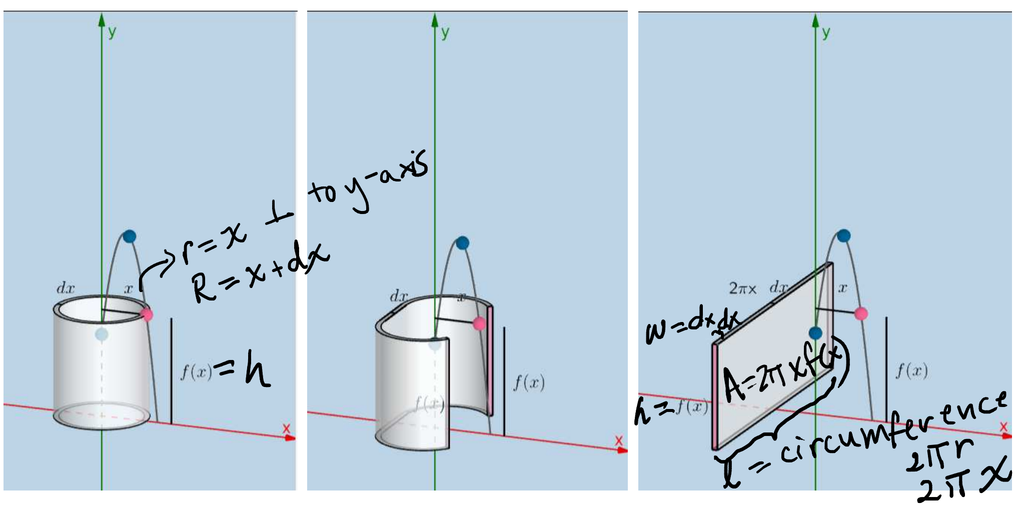


Pictures from Geogebra using shells:



When we found volume using washers, we integrated the area of a circle with a hole punched out of the middle. When we find the volume using shells, we integrate the area of a rectangle. The following pictures come from Geogebra and were created by Tim Brzezinski.





of the circle

$V_{shell} = \underbrace{2\pi x\, f(x)}_{\text{Area of the flattened rectangle}}\, dx$

What do we integrate?

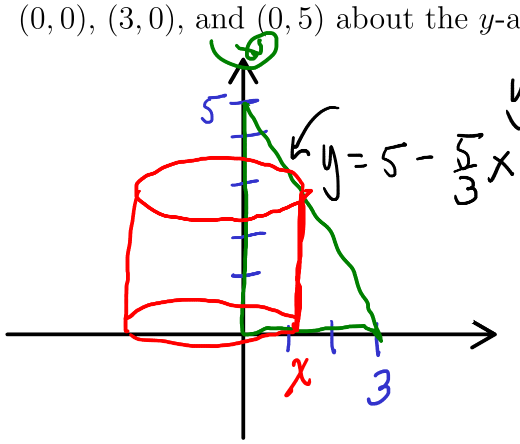
- **washers** - $\pi(R^2 - r^2)$
- **shells** - $2\pi rh$

What variable do we use to integrate?

Axis of Rotation	x-axis or horizontal axis	y-axis or vertical axis
washers	dx	dy
shells	dy	dx

IV Examples

Example 1. Use shells to find the volume of the solid obtained by rotating the triangle with vertices $(0,0)$, $(3,0)$, and $(0,5)$ about the y-axis.



y-axis – shells – dx

$r = x$

$h = 5 - \frac{5}{3}x$

$$V = \int_0^3 2\pi x \left(5 - \frac{5}{3}x\right) dx$$

$$= 2\pi \int_0^3 \left(5x - \frac{5}{3}x^2\right) dx$$

$$= 2\pi \left[\frac{5x^2}{2} - \frac{5x^3}{9} \Big|_0^3 \right]$$

$$= 2\pi \left[\left(\frac{45}{2} - 15\right) - (0 - 0) \right]$$

$$= 2\pi \left[\frac{15}{2} \right]$$

$$= 15\pi$$

right answer ☺

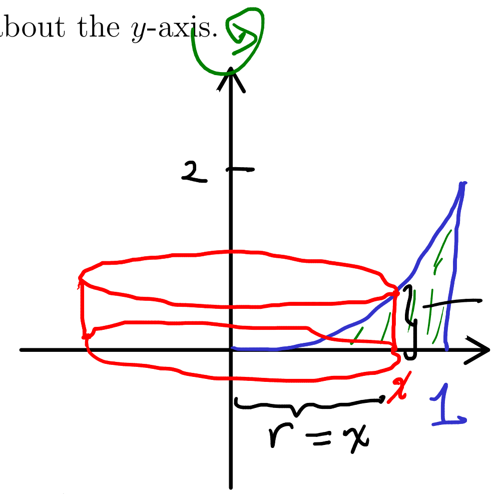
$\frac{1}{3}\pi (3)^2 (5) = 15\pi$

Example 2. Find the volume of the solid obtained by revolving the region enclosed by the curves

$$y = 2x^5, \quad \underbrace{y = 0}_{x\text{-axis}}, \quad \text{and } x = 1$$

about the y-axis.

y-axis - shells - dx



Shells

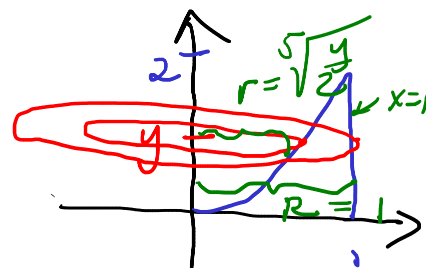
$$V = \int_0^1 2\pi x \, (2x^5) \, dx$$

$$= 4\pi \int_0^1 x^6 \, dx$$

$$= 4\pi \frac{x^7}{7} \Big|_0^1 = \frac{4\pi}{7}$$

washers.

y-axis - washers - dy



$y = 2x^5$

$\frac{y}{2} = x^5$

$x = \sqrt[5]{\frac{y}{2}}$

$$V = \int_0^2 \pi \left[1^2 - \left(\sqrt[5]{\frac{y}{2}} \right)^2 \right] dy$$

$$= \pi \int_0^2 \left(1 - \left(\frac{y}{2}\right)^{2/5} \right) dy$$

$u = \frac{y}{2}$

$$= \pi \left[y - 2 \cdot \frac{5}{7} \left(\frac{y}{2}\right)^{7/5} \Big|_0^2 \right]$$

$$= \frac{4\pi}{7}$$

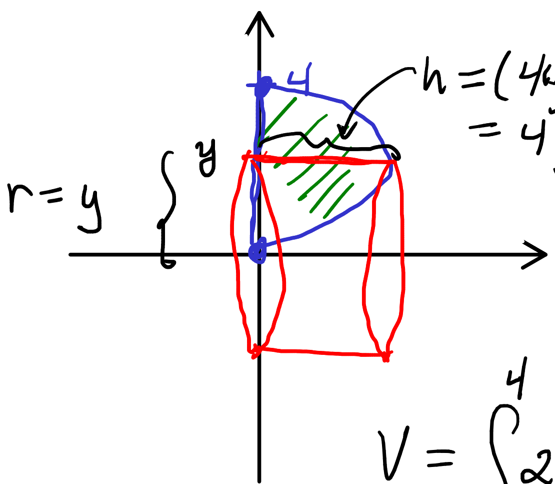
Example 3. Find the volume of the solid obtained by revolving the region enclosed by the curves

$$\underbrace{x = 0}_{y\text{-axis}} \quad \text{and } \underbrace{x = 4y - y^2}_{\substack{\text{parabola} \\ \text{opens left}}}$$

about the x-axis.

$x = y(4-y)$

y-int: $(0,0)$
$(0,4)$



$h = (4y - y^2) - 0$
$= 4y - y^2$

x-axis - shells - dy

$$V = \int_0^4 2\pi y(4y - y^2)\, dy$$

$$= 2\pi \int_0^4 (4y^2 - y^3)\, dy$$

$$= 2\pi \left[\frac{4y^3}{3} - \frac{y^4}{4} \Big|_0^4 \right]$$

$$= 2\pi \left[\frac{256}{3} - \frac{256}{4} \right]$$

$$= 2\pi \left[\frac{256}{12} \right]$$

$$= \frac{256\pi}{\cancel{6}}$$

$$= \frac{128\pi}{3}$$

Example 4. Find the volume of the solid obtained by revolving the region enclosed by the curves

$$\underbrace{x = 3 - y}_{\text{line}} \quad \text{and} \quad \underbrace{x = 3 + 2y - y^2}_{\text{parabola opens left}}$$

about the x-axis.

intersection pts

$3 - y = 3 + 2y - y^2$

$3 - y - 3 - 2y + y^2 = 0$

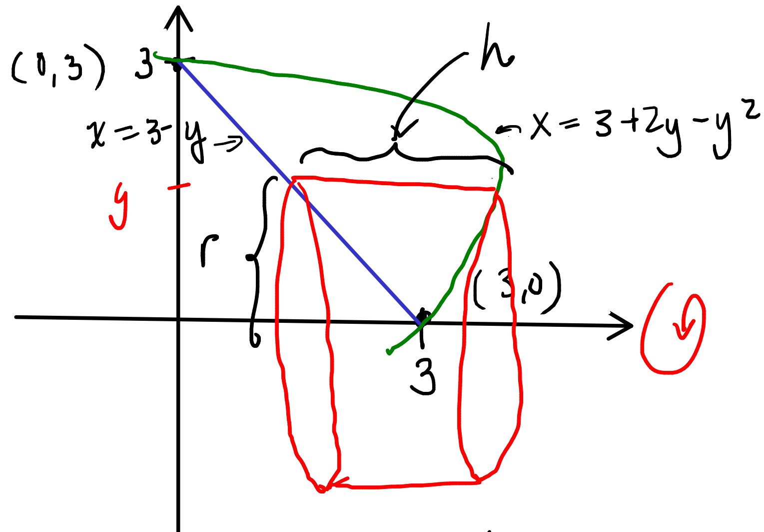
$y^2 - 3y = 0$

$y(y-3) = 0$

$y = 0 \qquad y = 3$

$x = 3 - 0 = 3 \qquad x = 3 - 3 = 0$

$(3,0) \qquad (0,3)$



$r = y \qquad h = (3+2y-y^2) - (3-y)$

Shells · ⟲ x-axis horizontal   use dy

$$V = \int_0^3 2\pi y\left((3+2y-y^2) - (3-y)\right)\, dy$$

$$= \int_0^3 2\pi y\,(3y - y^2)\, dy = 2\pi \int_0^3 (3y^2 - y^3)\, dy$$

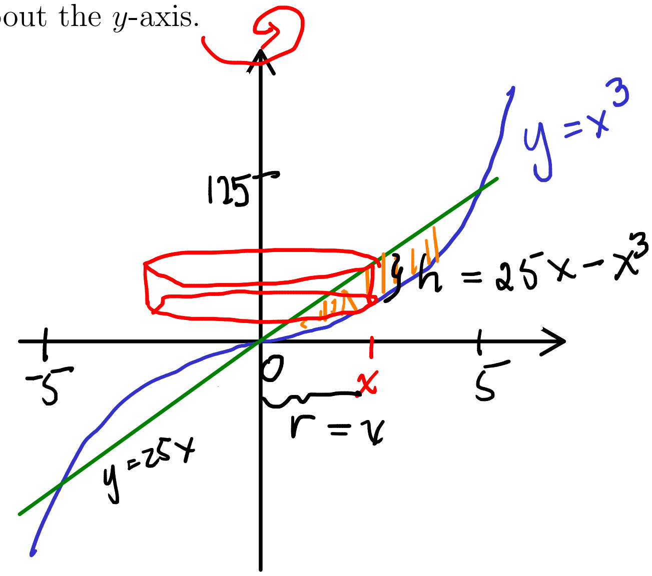
power rule

$$= \frac{27\pi}{2}$$

Example 5. Find the volume of the solid obtained by revolving the region enclosed in the first quadrant by the curves

$$y = x^3 \quad \text{and } y = 25x$$

about the y-axis.



intersection pts

$x^3 = 25x$

$x^3 - 25x = 0$

$x(x^2 - 25) = 0$

$x(x+5)(x-5) = 0$

$x = 0$, $y = 0$, $(0,0)$

$\underbrace{x = -5}_{\text{not within orange region}}$

$x = 5$, $y = 125$, $(5, 125)$

y-axis, shells, dx

$$V = \int_0^5 2\pi x\,(25x - x^3)\,dx = \text{distribute, power rule} = \frac{2500}{3}\pi$$

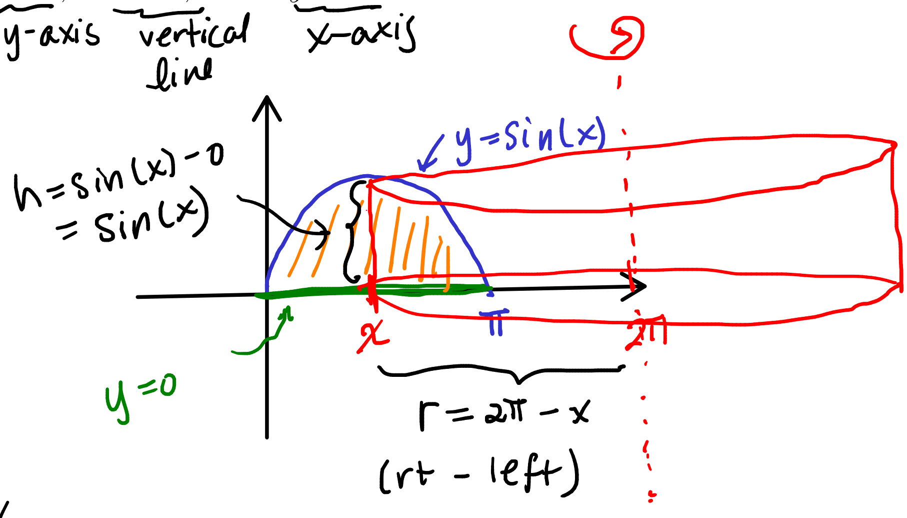
Example 6. Set up but **DO NOT EVALUATE** an integral expression for finding the volume of the solid obtained by revolving the region $\underbrace{\text{enclosed}}_{\text{bounded by}}$ by the curves

$$y = \sin(x), \quad \underbrace{x = 0}_{y\text{-axis}}, \quad \underbrace{x = \pi}_{\text{vertical line}}, \quad \text{and } \underbrace{y = 0}_{x\text{-axis}}$$

about . . .

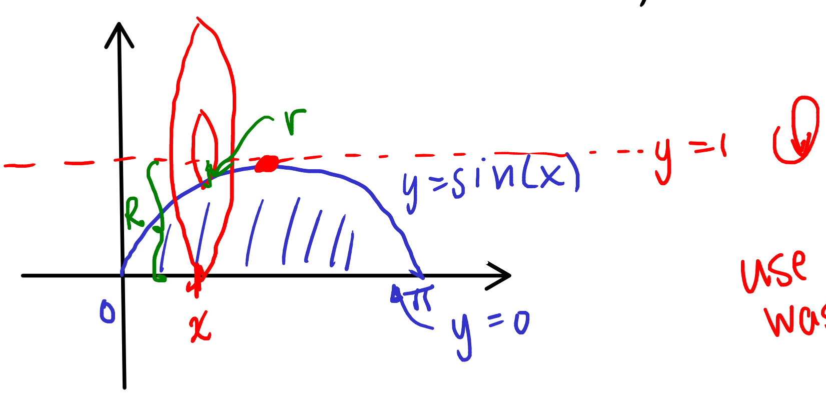
(a) the line $x = 2\pi$

vertical axis (like y-axis) - shells - use dx



$$V = \int_0^{\pi} 2\pi \underbrace{(2\pi - x)}_{r} \underbrace{(\sin(x))}_{h} \, dx$$

(b) the line $y = 1$ horizontal axis, like the x-axis



shells h - horizontal
easier w/ x = ___

washers radii - vertical
easier w/ y = ___

use washers

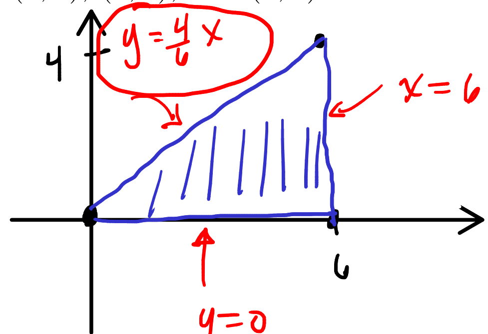
washers - horizontal axis - use dx
like x-axis

$R = 1 - 0 = 1$
$r = 1 - \sin(x)$

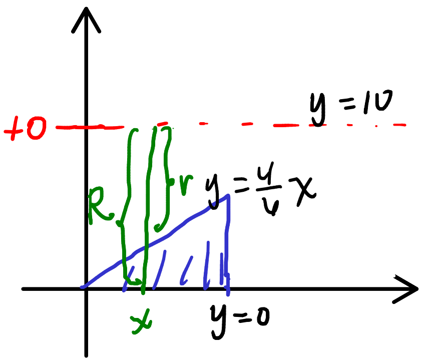
$$V = \int_0^{\pi} \pi \left[(1)^2 - (1 - \sin(x))^2 \right] dx$$

Example 7. Set up but **DO NOT EVALUATE** an integral expression for finding the volume of the solid obtained by revolving the region of the triangle with vertices $(0,0)$, $(6,4)$, and $(6,0)$ about . . .

$y = \frac{4}{6}x$ or $y = \frac{2}{3}x$ OR $x = \frac{3}{2}y$



(a) the line $y = 10$



horizontal axis, radii vertical, I have $y = \_\_$, use washers

washers, horizontal axis, use dx

$$V = \int_0^6 \pi\left[10^2 - \left(10 - \frac{4}{6}x\right)^2\right] dx$$

(b) the line $y = -2$

use washers:

$$V = \int_0^6 \pi\left[\left(\frac{4}{6}x + 2\right)^2 - 2^2\right] dx$$

$-(-2)$

(c) the line $x = 8$

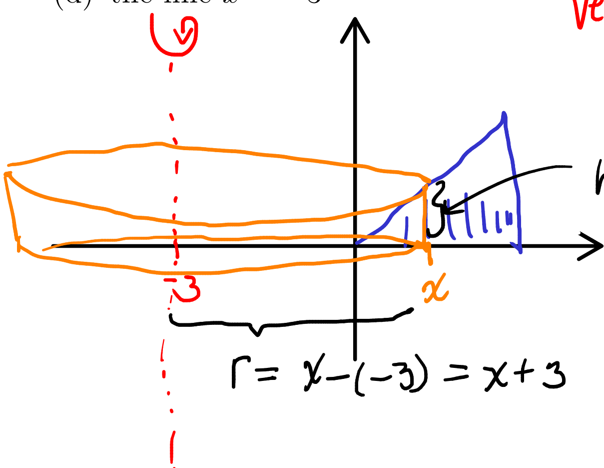
$$V = \int_0^6 2\pi(8 - x)\left(\frac{2}{3}x\right) dx$$

(d) the line $x = -3$

vertical axis, vertical heights, $y = \_\_$
use shells

$\Rightarrow$ use dx

$h = \frac{4}{6}x - 0 = \frac{4}{6}x$



$$V = \int_0^6 2\pi(x+3)\left(\frac{4}{6}x\right) dx$$